

# On the speed of convergence to stationarity via spectral gap: queueing networks with breakdowns and repairs

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## Abstract

We show that for unreliable Jackson networks spectral gap is strictly positive iff the spectral gaps for the corresponding coordinate birth and death processes are positive. Moreover, positivity of these spectral gaps is equivalent to the condition that for the stationary distribution of the network each of the one dimensional marginal distributions is strongly light-tailed, that is its discrete hazard function is separated from zero. Consequently, we obtain exponential ergodicity of unreliable networks iff for the corresponding stationary distribution the strong light-tailness condition holds. Connecting marginal distributions with the corresponding service rate functions, we characterize exponential rate of convergence in terms of the service rate functions.

Keywords: unreliable Jackson network; spectral gap; exponential ergodicity; Cheeger constant 60K25, 60J25

## 1 Introduction

We consider a class of Markov processes which describe Jackson type queueing networks with breakdowns and repairs. We show sufficient and necessary conditions for the existence of the spectral gap for some processes of this type, utilizing Cheeger's technical approach. As a consequence we obtain necessary and sufficient conditions for geometric ergodicity in  $L_2$  norm and in the total variation norm for such processes. These conditions are given in terms of hazard functions of the corresponding stationary distribution, and further in terms of the service rates in networks. More precisely, for unreliable Jackson networks spectral gap is strictly positive if and only if its stationary distribution is a product of strongly light-tailed (i.e. with the corresponding discrete hazard functions separated from 0) distributions. There is surely no spectral gap for the

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network when at least one of these distributions is not strongly light-tailed (usual light-tailness is not sufficient). It is worth mentioning that in the general theory of Markov processes such a property is not always present. There exist Markov processes with positive spectral gap, and having heavy-tailed stationary distributions. On the other hand there exist Markov processes without positive spectral gap and having light-tailed stationary distributions.

The problem of stability and ergodicity of queueing networks is a classical one and has been considered by many authors. Baccelli and Foss [2] considered quite general Jackson-type queueing networks, under stationarity and ergodicity assumptions on input sequences, and gave a path-wise construction of networks leading via subadditive ergodic theorem to some stability results for networks. Baccelli, Foss and Mairesse [3] provide interesting examples of networks with multiple stationary solutions. Another approach to stability of Jackson-type networks, namely by stochastic dominance, was given by Chang, Thomas and Kiang [4].

For networks which have unique stationary and ergodic solution one of the main problems is to estimate the probability of overflow. This problem is intimately connected with the transient behavior of networks before attaining stationarity and with the question about the speed of convergence to stationarity. Large deviations is a natural technique to deal with overflow probabilities. The problem of large deviations for classical Jackson networks was analyzed in detail by Ignatiouk-Robert [14], using so-called twisted kernel which involve exponential change of measure, and showing explicitly the rate function for a two nodes network. Foley and McDonald [11] show large deviations for a two nodes network with interaction and show some rough asymptotics for stationary distribution of such process. Adan, Foley and McDonald [1] developed a method for exact asymptotics for stationary distribution of Markov chain, which also can be applied to modified Jackson networks. Lorek [20] gives exact asymptotics for stationary distribution for some modified networks. Another method to deal with overflow probabilities is simulation. Dupuis and Wang [9] developed simple and efficient state-dependent importance sampling schemes for simulating buffer overflows in stable Jackson networks. The mentioned results on overflow are directly related to one of the key performance properties in queueing systems, namely to the exponential decay rate of the steady-state tail probabilities of the queue lengths. It is of course present for classical queueing networks with constant service rates but in a more general setting it is not always present. For example, for multi-type networks it is known that if the stochastic primitives have finite moments, then the queue lengths also have finite moments, and the tail probability of the queue length decays faster than any polynomial. However, it is not always decaying exponentially fast even with exponential service times, as shown by Gamarnik and Meyn [12]. The rate of exponential decay of the steady-state tail probabilities of the queue lengths was studied also by Kroese, Scheinhardt and Taylor [17] who interpreted Jackson tandem networks as a quasi birth and death (QBD) process and studied relations of the decay rates to the convergence norm of Neuts'  $R$  matrix. Dieker and Warren [8], using an approach via non-crossing probabilities, showed among other things that for the classical tandem Jackson networks its spectral gap is equal to the spectral gap of the  $M/M/1$  queue with the same arrival and service rates as the network's bottleneck station, which allows to relate the corresponding relaxation times with the speed of convergence to stationarity.

Fayolle et al. [10] considered the classical Jackson networks with constant service rates for each station. They constructed explicitly Lyapounov functions for these networks and showed the necessary and sufficient conditions for ergodicity without using the classical Jackson's product

form. Moreover they showed an exponential rate of convergence in  $L_1$  norm to the invariant distribution of the corresponding Markov process. Hordijk and Spieksma [13] showed a similar result, obtaining geometric rate of convergence, but measured in the so called  $f$ -norm. An alternative measure for the convergence to stationarity of a stochastically increasing Markov chains can be expressed in terms of the deviation matrices as presented for  $M/M/s/N$  and  $M/M/s/\infty$  queues by Koole and Spieksma [16], and further by Coolen-Schrijner and Van Doorn [6] for general ergodic Markov chains.

The approach via spectral gap to the speed of convergence is a classical one in the context of particle systems, see e.g. Liggett [19]. The problem of existence of the spectral gap for Jackson networks with non-constant service rates  $\mu_i(\cdot)$  was considered by Iscoe and McDonald [15]. They showed that if a stationary distribution is a product of strongly light-tailed distributions then the spectral gap exists.

We consider another modification of classical Jackson networks, allowing unreliable nodes. Unreliable Jackson networks are networks, where in some subsets of the set of nodes the service stations can be broken and then repaired during the time evolution of the system. The breakdown and repair events can be of a rather general nature, but driven by a Markov process. In the time intervals when nodes are broken, there are several rules for rerouting. For full details of such networks see Sauer and Daduna [23], and Sauer [22]. In the latter also a rich historical overview of such networks can be found. Sauer in [22] showed that in the case of constant service rates  $\mu_i(\cdot) = \mu_i$  and without rerouting (i.e. customers are allowed to join the queue at broken server, where they wait till repair), the corresponding discrete time uniformization chain is a Lyapounov Markov chain according to the terminology used by Fayolle *et al.* [10]. Consequently, for such networks the rate of convergence in total variation distance is exponential and the stationary distribution is light tailed.

We assume in our spectral gap result for unreliable networks that the routing matrix is reversible. For classical networks the reversibility assumption can be skipped. In the proof we use some ideas from the mentioned paper by Iscoe and McDonald [15], where the authors studied exit times for the classical Jackson networks, and they showed the existence of the spectral gap under the assumption of strong light-tailness - using some auxiliary birth and death processes. In our proof we relate the network process to the independent copies of the coordinate birth and death processes directly, so technically taken our proof is a different one. Stationary distribution of a Jackson network has a remarkable property: it is of a product form. Sauer and Daduna [23] showed an analogous property for the stationary distribution for unreliable Jackson networks. The product form of the stationary distribution for unreliable networks is a crucial property used in our work to extend the results of Iscoe and McDonald [15], and of Sauer [22]. Using the existence of the spectral gap for unreliable queueing networks, we formulate an *if and only if* result connecting the exponential speed of convergence to stationarity with the strong light-tailness property of the stationary distribution.

The task of obtaining positive lower bounds for the spectral gap of Jackson networks with state dependent service rates involves bounds on the spectral gaps for the corresponding coordinate birth and death processes (as in this paper) or some other related birth and death processes (as in Iscoe and McDonald [15]), and some additional factors reflecting dependence among the nodes of the network. The problem of bounding the rates of convergence of birth-death processes is a classical one, see e.g. Van Doorn [24] for a summary. A related comparison result for spectral

gaps for networks is given by Daduna and Szekli [7], Proposition 3.6, where a direct comparison to the spectral gaps for the corresponding coordinate birth and death processes is given without additional factors, but using an additional assumption on the routing. As an illustration we shall give some bounds on the spectral gap of a Jackson network in the last section, however the problem of finding exact bounds for (unreliable) networks is not solved in a satisfactory form and we leave it as a future research topic.

The paper is organized as follows. In the next section we introduce unreliable networks by giving the respective generator. In section 3 we give the result relating the existence of the spectral gap of unreliable networks with the tail properties of its stationary distribution. We give a proof of this result and formulate direct consequences for exponential ergodicity of the network process. In section 4 we use equilibrium rates to reformulate our results from section 3. We show that strong light-tailness is stronger than usual light-tailness. Finally we give some bounds on the spectral gap for a Jackson network in section 4.2.

## 2 Definition of the process

The classical **Jackson network** consists of  $m$  numbered servers, denoted by  $M := \{1, \dots, m\}$ . Station  $j \in M$  is a single server queue with infinite waiting room under FCFS (First Come First Served) regime. All the customers in the network are indistinguishable. There is an external Poisson arrival stream with intensity  $\lambda$  and arriving customers are sent to node  $j$  with probability  $r_{0j}$ ,  $\sum_{j=1}^m r_{0j} = r \leq 1$ . Customers arriving at node  $j$  from the outside or from other nodes request a service which is at node  $j$  provided with intensity  $\mu_j(n)$  ( $\mu_j(0) := 0$ ), where  $n$  is the number of customers at node  $j$  including the one being served. All the service times and arrival processes are assumed to be independent.

A customer departing from node  $i$  immediately proceeds to node  $j$  with probability  $r_{ij} \geq 0$  or departs from the network with probability  $r_{i0}$ . The routing is independent of the past of the system given the momentary node where the customer is. We assume that the matrix  $R := (r_{ij}, i, j \in M)$  is irreducible.

Let  $X_j(t)$  be the number of customers present at node  $j$ , at time  $t \geq 0$ . Then

$$X(t) = (X_1(t), \dots, X_m(t))$$

is the joint queue length vector at time instant  $t \geq 0$  and  $\mathbf{X} := (X(t), t \geq 0)$  is the joint queue length process with the state space  $\mathbb{E} = \mathbb{Z}_+^m$ .

The unique stationary distribution for  $\mathbf{X}$  exists if and only if the unique solution of the **traffic equation**

$$\lambda_i = \lambda r_{0i} + \sum_{j=1}^m \lambda_j r_{ji}, \quad i = 1, \dots, m \quad (1)$$

satisfies

$$C_i := 1 + \sum_{n=1}^{\infty} \frac{\lambda_i^n}{\prod_{y=1}^n \mu_i(y)} < \infty, \quad 1 \leq i \leq m.$$

The parameters of a Jackson network are: the arrival intensity  $\lambda$ , the routing matrix  $R$  (with its traffic vector  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$ ), the vector of service rates  $\boldsymbol{\mu} = (\mu_1(\cdot), \dots, \mu_m(\cdot))$  and the number of servers  $m$ . Our standing assumption for all considered networks is that for all  $j$ ,  $\underline{\mu}_j := \inf_n \mu_j(n) > 0$ . We denote the overall minimal service intensity by  $\underline{\mu} = \min_j \underline{\mu}_j$ .

Assume now that the servers at the nodes in the Jackson network are unreliable, i.e., the nodes may break down. The breakdown event may occur in different ways. Nodes may break down as an isolated event or in groups simultaneously, and the repair of the nodes may end for each node individually or in groups as well. It is not required that those nodes which stopped service simultaneously return to service at the same time instant. To describe the system's evolution we have to enlarge the state space for the network process as it will be described below. Denote by  $M_0 := \{0, 1, \dots, m\}$  the set of nodes enlarged by adding the *outside* node.

- Let  $D \subset M$  be the set of servers out of order, i.e. in *down status* and  $I \subset M \setminus D, I \neq \emptyset$  be the subset of nodes in *up status*. Then the servers in  $I$  break down with intensity  $\alpha_{D \cup I}^D(n_i : i \in M)$ .
- Let  $D \subset M$  be the set of servers in *down status* and  $H \subset D, H \neq \emptyset$ . The broken servers from  $H$  return from repair with intensity  $\beta_{D \setminus H}^D(n_i : i \in M)$ .
- The routing is changed according to so-called REPETITIVE SERVICE - RANDOM DESTINATION BLOCKING (RS-RD BLOCKING) rule: For  $D$  - set of servers under repair routing probabilities are restricted to nodes from  $M_0 \setminus D$  as follows:

$$r_{ij}^D = \begin{cases} r_{ij}, & i, j \in M_0 \setminus D, \quad i \neq j, \\ r_{ii} + \sum_{k \in D} r_{ik}, & i \in M_0 \setminus D, \quad i = j. \end{cases}$$

The external arrival rates are

$$\lambda r_{0j}^D = \lambda r_{0j} \text{ for nodes } j \in M \setminus D, \quad (2)$$

and zero, otherwise. Let  $R^D = (r_{ij}^D)_{i,j \in M_0 \setminus D}$  be the modified routing. Note that  $R^\emptyset = R$ .

We assume for the intensities of breakdowns and repairs  $\emptyset \neq I \subset D$  and  $\emptyset \neq H \subset M \setminus D$  that

$$\alpha_{D \cup I}^D(n_i : i \in M) := \frac{\psi(D \cup I)}{\psi(D)},$$

$$\beta_{D \setminus H}^D(n_i : i \in M) := \frac{\phi(D)}{\phi(D \setminus H)},$$

where  $\psi$  and  $\phi$  are arbitrary positive functions defined for all subsets of the set of nodes. That means that breakdown and repair intensities depend on the sets of servers but are independent of the particular numbers of customers present in these servers.

In order to describe unreliable Jackson networks we need to attach to the state space  $\mathbb{Z}_+^m$  of the corresponding standard network process an additional component which includes information of availability of the system. We consider new state space

$$\tilde{\mathbf{n}} = (D, n_1, n_2, \dots, n_m) \in \mathcal{P}(M) \times \mathbb{Z}_+^m =: \tilde{\mathbb{E}},$$

where  $\mathcal{P}(M)$  denotes the powerset of  $M$ . The first coordinate in  $\mathbf{n}$  we call the availability coordinate.

The set  $D$  is the set of servers in *down status*. At node  $i \in D$  there are  $n_i$  customers waiting for server being repaired. Denote possible transitions by

$$\begin{aligned} T_{ij}\tilde{\mathbf{n}} &:= (D, n_1, \dots, n_i - 1, \dots, n_j + 1, \dots, n_m), \\ T_{0j}\tilde{\mathbf{n}} &:= (D, n_1, \dots, n_j + 1, \dots, n_m), \\ T_{i0}\tilde{\mathbf{n}} &:= (D, n_1, \dots, n_i - 1, \dots, n_m), \\ T_H\tilde{\mathbf{n}} &:= (D \setminus H, n_1, \dots, n_m), \\ T^I\tilde{\mathbf{n}} &:= (D \cup I, n_1, \dots, n_m). \end{aligned} \tag{3}$$

**Definition 2.1.** The Markov process  $\tilde{\mathbf{X}} = (\tilde{X}(t), t \geq 0)$  defined by the infinitesimal generator

$$\begin{aligned} \tilde{\mathbf{Q}}f(\tilde{\mathbf{n}}) = & \sum_{j=1}^m [f(T_{0j}\tilde{\mathbf{n}}) - f(\tilde{\mathbf{n}})]\lambda r_{0j}^D + \sum_{i=1}^m \sum_{j=1}^m [f(T_{ij}\tilde{\mathbf{n}}) - f(\tilde{\mathbf{n}})]\mu_i(n_i)r_{ij}^D + \\ & \sum_{\substack{I \subset M \\ |I|=m}} [f(T^I\tilde{\mathbf{n}}) - f(\tilde{\mathbf{n}})] \frac{\psi(D \cup I)}{\psi(D)} + \sum_{H \subset M} [f(T_H\tilde{\mathbf{n}}) - f(\tilde{\mathbf{n}})] \frac{\phi(D)}{\phi(D \setminus H)} + \\ & \sum_{j=1}^m [f(T_{j0}\tilde{\mathbf{n}}) - f(\tilde{\mathbf{n}})]\mu_j(n_j)r_{j0}^D \end{aligned} \tag{4}$$

is called **unreliable Jackson network**.

We denote the corresponding transition intensities (written in a matrix form) by  $[\tilde{q}(\tilde{\mathbf{n}}, \tilde{\mathbf{n}}')]_{\tilde{\mathbf{n}}, \tilde{\mathbf{n}}' \in \tilde{\mathbb{E}}}$ . Similarly to the classical case the invariant distribution for this Markov process can be written in a product form.

**Theorem 2.2** (Sauer and Daduna [23]). *Let  $\tilde{\mathbf{X}}$  be unreliable Jackson network following the RS-RD-BLOCKING. If the routing matrix  $R$  is reversible, i.e.:*

$$\lambda_j r_{ji} = \lambda_i r_{ij}, \quad i, j \in M_0,$$

*then the stationary distribution of process  $\tilde{\mathbf{X}}$  is given by*

$$\pi(\tilde{\mathbf{n}}) = \pi(D, n_1, \dots, n_m) = \frac{1}{C} \frac{\psi(D)}{\phi(D)} \prod_{i=1}^m \pi_i(n_i), \tag{5}$$

where

$$\pi_i(n_i) = \frac{1}{C_i} \frac{\lambda_i^{n_i}}{\prod_{k=1}^{n_i} \mu_i(k)}, \quad C_i = \sum_{n=0}^{\infty} \frac{\lambda_i^n}{\prod_{y=1}^n \mu_i(y)} \tag{6}$$

and  $C$  is the normalization constant used for the availability coordinate.

Constants  $C_i, i = 1, \dots, m$  are all finite if and only if the network is ergodic.

Note that in this generality, the vector of the number of customers alone, without the availability coordinate, does not form a Markov process. It is a Markov process if the availability coordinate process is a trivial one, that is if nothing breaks down all the time. In this case this Markov process is identical with the classical Jackson network, and the reversibility assumption on routing is not needed then in order to obtain the classical product formula.

### 3 Spectral gap and convergence

Consider a pure jump Markov process  $\mathbf{X} = (X_t, t \geq 0)$  with denumerable state space  $\mathbb{E}$ , with the matrix of transition intensities  $Q = [q(\mathbf{e}, \mathbf{e}')]_{\mathbf{e}, \mathbf{e}' \in \mathbb{E}}$ , stationary distribution  $\pi$  and the corresponding infinitesimal generator  $\mathbf{Q}$  given by

$$\mathbf{Q}f(\mathbf{e}) = \sum_{\mathbf{e}' \in \mathbb{E}} [f(\mathbf{e}') - f(\mathbf{e})]q(\mathbf{e}, \mathbf{e}'), \quad \mathbf{e} \in \mathbb{E}.$$

The usual scalar product on  $L_2(\mathbb{E}, \pi)$  is denoted by

$$\langle f, g \rangle_\pi = \sum_{\mathbf{e} \in \mathbb{E}} f(\mathbf{e})g(\mathbf{e})\pi(\mathbf{e}), \quad \|f\|_2^2 = \langle f, f \rangle_\pi,$$

and by  $\mathbf{1}$  the constant function equal to 1. We shall use the symbol  $\pi(f)$  to denote  $\langle f, \mathbf{1} \rangle_\pi$ .

We denote the spectral gap for  $\mathbf{X}$  by

$$\text{Gap}(\mathbf{Q}) := \inf \{ -\langle f, \mathbf{Q}f \rangle_\pi : \|f\|_2 = 1, \pi(f) = 0 \}. \quad (7)$$

The main result on the spectral gap for networks is a consequence of the following lemma, which relates the networks to the *corresponding coordinate birth and death processes*. The corresponding birth and death processes are independent processes which have their invariant distributions equal to  $\pi_i$  distributions in the product form formula for networks' stationary distribution.

**Lemma 3.1.** *Let  $\tilde{\mathbf{X}}$  be unreliable Jackson network process following the RS-RD- BLOCKING, with the infinitesimal generator  $\tilde{\mathbf{Q}}$ . Suppose that  $\tilde{\mathbf{Q}}$  is bounded, and the minimal service intensity  $\underline{\mu} > 0$ . If the routing matrix  $R$  is reversible and regular ( $R^k > 0$  for some  $k \geq 1$ ) then  $\text{Gap}(\tilde{\mathbf{Q}}) > 0$  if and only if for each  $i = 1, \dots, m$ , the birth and death process with constant birth rates  $\lambda_i$  and state dependent death rates  $\mu_i(n)$  have their corresponding spectral gaps positive.*

**Theorem 3.2.** *Let  $\tilde{\mathbf{X}}$  be unreliable Jackson network process following the RS-RD- BLOCKING, with the infinitesimal generator  $\tilde{\mathbf{Q}}$ . Suppose that  $\tilde{\mathbf{Q}}$  is bounded and the minimal service intensity  $\underline{\mu} > 0$ . If the routing matrix  $R$  is reversible, and regular then  $\text{Gap}(\tilde{\mathbf{Q}}) > 0$  if and only if for each  $i = 1, \dots, m$ , distribution  $(\pi_i)_{i \geq 0}$  is strongly light-tailed, i.e.  $\inf_k h_{\pi_i}(k) > 0$ , for  $h_{\pi_i}(k) = \frac{\pi_i(k)}{\sum_{j \geq k} \pi_i(j)}$ .*

The proof of this theorem will be given in an extra section after gathering some useful facts on birth and death processes in the next paragraph.

#### 3.1 Spectral gap for related birth and death processes

A proof of the next theorem can be found in Liggett [19], Theorem 3.7. The formulation of it is simplified to the case of state independent birth rates. For a more complete description of criteria for positivity of spectral gap for jump processes see the book by Mu-Fa Chen [5], chapter 5.

**Theorem 3.3** (Liggett [19]). Assume that  $\mathbf{Z}$  is a birth and death process on  $\mathbb{Z}_+$ , with state independent birth rates  $\lambda > 0$ , and possibly state dependent death rates  $\mu(n) > 0$ , and for all  $i \geq 0$ , and for some  $b, c > 0$ , we have

$$\sum_{j>i} \pi(j) \leq c\pi(i)\lambda \quad \text{and} \quad \sum_{j>i} \pi(j) \leq b\pi(i).$$

Then for the corresponding generator  $\mathbf{Q}$ ,

$$\text{Gap}(\mathbf{Q}) \geq \frac{(\sqrt{b+1} - \sqrt{b})^2}{c} \geq \frac{1}{2c(1+2b)}.$$

In the case of constant birth rates, from the Corollary 3.8 of Liggett [19], we have that a necessary and sufficient condition for  $\text{Gap}(\mathbf{Q})$  to be positive is that the stationary distribution is such that

$$\inf_i \frac{\pi(i)}{\sum_{j \geq i} \pi(j)} > 0.$$

It is natural to recall at this point that the ratio above defines so called *discrete hazard function*. For arbitrary distribution  $(p(i), i \geq 0)$  on  $\mathbb{Z}_+$ , such that for all  $i$ ,  $p(i) > 0$  we define its hazard function by

$$h_p(i) = \frac{p(i)}{\sum_{j \geq i} p(j)}.$$

This function has some analogous properties to the continuous hazard rate function and is convenient to describe tail properties of the corresponding distribution function. We will get back to this point in a later section. We say that a distribution  $(p(i), i \geq 0)$  on  $\mathbb{Z}_+$  is *strongly light-tailed* if

$$\inf_{i \geq 0} h_p(i) > 0.$$

We shall see later that such a distribution is light-tailed in a usual sense.

We shall use the following reformulation of Liggett's result

**Proposition 3.4.** Assume that  $\mathbf{Z}$  is a birth and death process on  $\mathbb{Z}_+$ , with state independent birth rates  $\lambda > 0$ , and possibly state dependent death rates  $\mu(n) > 0$ , and for all  $i \geq 0$ , and for some  $\epsilon > 0$ , uniformly in  $i$ , we have

$$h_\pi(i) \geq \epsilon.$$

Then for the corresponding generator  $\mathbf{Q}$ ,

$$\text{Gap}(\mathbf{Q}) \geq \frac{\lambda \epsilon^2}{2(1-\epsilon)(2-\epsilon)}.$$

A necessary and sufficient condition for  $\text{Gap}(\mathbf{Q})$  to be positive is that the stationary distribution  $\pi$  is strongly light-tailed.

In the proof of Theorem 3.2 we shall compare networks processes with their coordinate birth and death processes utilizing the following theorem.

**Theorem 3.5** (Liggett [19], Th. 2.6). *Suppose that a pure jump Markov process  $\mathbf{X}$ , with generator  $\mathbf{Q}$  and stationary distribution  $\pi$  evolves on the product state space  $\mathbb{E} = \mathbb{E}_1 \times \mathbb{E}_2 \times \cdots \times \mathbb{E}_m$ ,  $m \geq 1$ , having coordinates which are independent Markov processes such that  $i$ -th coordinate has generator  $\mathbf{Q}_i$ , denumerable state space  $\mathbb{E}_i$  and invariant probability measure  $\pi_i$ . Then  $\pi$  is the product measure of  $\pi_i$ 's and*

$$\text{Gap}(\mathbf{Q}) = \inf_i \text{Gap}(\mathbf{Q}_i).$$

### 3.2 Proof of the theorem on spectral gaps

We shall first proof Lemma 3.1.

Denote by  $\hat{\mathbf{Q}}$  the generator associated with  $(m+1)$ -dimensional vector  $(\mathbf{Y}_t, \hat{\mathbf{X}}_t)$ , where  $\hat{\mathbf{X}}_t$  is the vector of  $m$  independent birth and death processes with generators  $\hat{\mathbf{Q}}_i$ ,  $i = 1, \dots, m$ , given by

$$\hat{\mathbf{Q}}_i f(n) = [f(n+1) - f(n)]\lambda_i + [f(n) - f(n-1)]\mu_i(n), \quad n \in \mathbb{N}. \quad (8)$$

We write  $[\hat{q}(\tilde{\mathbf{n}}, \tilde{\mathbf{n}}')]_{\tilde{\mathbf{n}}, \tilde{\mathbf{n}}' \in \tilde{\mathbb{E}}}$  for the corresponding transition intensities.

The stationary distribution of the process with generator  $\hat{\mathbf{Q}}_i$  is  $\pi_i$ , which is given in the product formula (6) for networks. Let  $\mathbf{Y}_t$  be the process on state space  $\mathcal{P}(M)$  with infinitesimal generator denoted by  $\hat{\mathbf{Q}}_0$  and the stationary distribution:

$$\pi_0(I) = \frac{1}{C} \frac{\psi(I)}{\phi(I)}, \quad C := \left( \sum_{I \subset M} \frac{\psi(I)}{\phi(I)} \right).$$

From Theorem 3.5 we have that  $\text{Gap}(\hat{\mathbf{Q}}) = \min_{0 \leq i \leq m} \text{Gap}(\hat{\mathbf{Q}}_i)$ . The state space  $\mathcal{P}(M)$  is finite, thus  $\text{Gap}(\hat{\mathbf{Q}}_0) > 0$ . In order to compare spectral gaps of the process with the generator  $\hat{\mathbf{Q}}$  and the network process with the generator  $\tilde{\mathbf{Q}}$  we will use Cheeger's constants.

Define the following Cheeger's constants:

$$\begin{aligned} \kappa_1(A) &:= \frac{\sum_{\tilde{\mathbf{n}} \in A} \pi(\tilde{\mathbf{n}}) \tilde{q}(\tilde{\mathbf{n}}, A^c)}{\pi(A) \pi(A^c)}, & \kappa_1 &:= \inf_{A \subset \tilde{\mathbb{E}}, \pi(A) \in (0,1)} \kappa_1(A), \\ \kappa_2(A) &:= \frac{\sum_{\tilde{\mathbf{n}} \in A} \pi(\tilde{\mathbf{n}}) \hat{q}(\tilde{\mathbf{n}}, A^c)}{\pi(A) \pi(A^c)}, & \kappa_2 &:= \inf_{A \subset \tilde{\mathbb{E}}, \pi(A) \in (0,1)} \kappa_2(A), \end{aligned}$$

where  $\pi$  is taken from (5).

We shall show that there exist  $v_1 > 0, v_2 > 0$  such that uniformly for all  $A \subset \tilde{\mathbb{E}}$

$$v_2 \sum_{\tilde{\mathbf{n}} \in A} \pi(\tilde{\mathbf{n}}) \hat{q}(\tilde{\mathbf{n}}, A^c) \geq \sum_{\tilde{\mathbf{n}} \in A} \pi(\tilde{\mathbf{n}}) \tilde{q}(\tilde{\mathbf{n}}, A^c) \geq v_1 \sum_{\tilde{\mathbf{n}} \in A} \pi(\tilde{\mathbf{n}}) \hat{q}(\tilde{\mathbf{n}}, A^c). \quad (9)$$

Suppose that we show the existence of such  $v_1, v_2$  as in (9), and suppose that  $\text{Gap}(\hat{\mathbf{Q}}) > 0$ . From Theorem 2.1 in Lawler and Sokal [18], we have that  $0 < \text{Gap}(\hat{\mathbf{Q}}) \leq \kappa_2$ , and uniformly in  $A$ ,  $\kappa_2(A) \leq (v_1)^{-1} \kappa_1(A)$ , hence  $\kappa_1 > 0$ , which in turn using Theorem 2.3 in Lawler and Sokal [18]

(which assures that  $\kappa_1^2/(8|\tilde{\mathbf{Q}}|) \leq \text{Gap}(\tilde{\mathbf{Q}})$ ) implies that  $\text{Gap}(\tilde{\mathbf{Q}}) > 0$ . Here  $|\tilde{\mathbf{Q}}|$  is the supremum norm of the bounded generator  $\tilde{\mathbf{Q}}$ .

Similarly, it is possible to argue that  $\text{Gap}(\tilde{\mathbf{Q}}) > 0$  implies that  $\text{Gap}(\hat{\mathbf{Q}}) > 0$ .

In order to complete the proof we turn now to show validity of (9) which is equivalent to

$$\inf_{\substack{A \subset \tilde{\mathbb{E}} \\ \pi(A) \in (0,1)}} \left\{ \frac{\sum_{\tilde{\mathbf{n}} \in A} \pi(\tilde{\mathbf{n}}) \tilde{q}(\tilde{\mathbf{n}}, A^c)}{\sum_{\tilde{\mathbf{n}} \in A} \pi(\tilde{\mathbf{n}}) \hat{q}(\tilde{\mathbf{n}}, A^c)} \right\} > 0 \quad \text{and} \quad \sup_{\substack{A \subset \tilde{\mathbb{E}} \\ \pi(A) \in (0,1)}} \left\{ \frac{\sum_{\tilde{\mathbf{n}} \in A} \pi(\tilde{\mathbf{n}}) \tilde{q}(\tilde{\mathbf{n}}, A^c)}{\sum_{\tilde{\mathbf{n}} \in A} \pi(\tilde{\mathbf{n}}) \hat{q}(\tilde{\mathbf{n}}, A^c)} \right\} < \infty. \quad (10)$$

We have

$$\frac{\sum_{\tilde{\mathbf{n}} \in A} \pi(\tilde{\mathbf{n}}) \tilde{q}(\tilde{\mathbf{n}}, A^c)}{\sum_{\tilde{\mathbf{n}} \in A} \pi(\tilde{\mathbf{n}}) \hat{q}(\tilde{\mathbf{n}}, A^c)} = \frac{\sum_{\tilde{\mathbf{n}} \in A} \pi(\tilde{\mathbf{n}}) \left( \sum_{T_{ij} \tilde{\mathbf{n}} \in A^c} \mu_i(n_i) r_{ij}^D + \sum_{T_{0j} \tilde{\mathbf{n}} \in A^c} \lambda r_{0j} + \sum_{T_{i0} \tilde{\mathbf{n}} \in A^c} \mu_i(n_i) r_{i0}^D + \sum_{T^I \tilde{\mathbf{n}} \in A^c} \frac{\psi(D \cup I)}{\psi(D)} + \sum_{T_H \tilde{\mathbf{n}} \in A^c} \frac{\phi(D)}{\phi(D \setminus H)} \right)}{\sum_{\tilde{\mathbf{n}} \in A} \pi(\tilde{\mathbf{n}}) \left( \sum_{T_{i0} \tilde{\mathbf{n}} \in A^c} \mu_i(n_i) + \sum_{T_{0j} \tilde{\mathbf{n}} \in A^c} \lambda_j + \sum_{T^I \tilde{\mathbf{n}} \in A^c} \frac{\psi(D \cup I)}{\psi(D)} + \sum_{T_H \tilde{\mathbf{n}} \in A^c} \frac{\phi(D)}{\phi(D \setminus H)} \right)}.$$

Let

$$\tilde{q}^{\min} = \inf_{A, \tilde{\mathbf{n}} \in A: \tilde{q}(\tilde{\mathbf{n}}, A^c) > 0} \{ \tilde{q}(\tilde{\mathbf{n}}, A^c) \}, \quad \tilde{q}^{\max} = \sup_{A, \tilde{\mathbf{n}} \in A: \tilde{q}(\tilde{\mathbf{n}}, A^c) > 0} \{ \tilde{q}(\tilde{\mathbf{n}}, A^c) \}.$$

From our assumptions the generators are bounded and  $\underline{\mu} > 0$ , therefore  $\tilde{q}^{\min} > 0$ , and  $\tilde{q}^{\max} < \infty$ .

For

$$\hat{q}^{\min} = \inf_{A, \tilde{\mathbf{n}} \in A: \hat{q}(\tilde{\mathbf{n}}, A^c) > 0} \{ \hat{q}(\tilde{\mathbf{n}}, A^c) \}, \quad \hat{q}^{\max} = \sup_{A, \tilde{\mathbf{n}} \in A: \hat{q}(\tilde{\mathbf{n}}, A^c) > 0} \{ \hat{q}(\tilde{\mathbf{n}}, A^c) \},$$

we also have  $\hat{q}^{\min} > 0$  and  $\hat{q}^{\max} < \infty$ .

For a fixed  $A$  it is natural to define

$$\partial \tilde{A} = \{ \tilde{\mathbf{n}} \in A : \tilde{q}(\tilde{\mathbf{n}}, A^c) > 0 \}, \quad \partial \hat{A} = \{ \tilde{\mathbf{n}} \in A : \hat{q}(\tilde{\mathbf{n}}, A^c) > 0 \}.$$

Because the ratio under consideration can be written as

$$\frac{\sum_{\tilde{\mathbf{n}} \in \partial \tilde{A}} \pi(\tilde{\mathbf{n}}) \tilde{q}(\tilde{\mathbf{n}}, A^c)}{\sum_{\tilde{\mathbf{n}} \in \partial \hat{A}} \pi(\tilde{\mathbf{n}}) \hat{q}(\tilde{\mathbf{n}}, A^c)}$$

we obtain

$$\frac{\tilde{q}^{\max}}{\hat{q}^{\min}} \cdot \frac{\sum_{\tilde{\mathbf{n}} \in \partial \tilde{A}} \pi(\tilde{\mathbf{n}})}{\sum_{\tilde{\mathbf{n}} \in \partial \hat{A}} \pi(\tilde{\mathbf{n}})} \geq \frac{\sum_{\tilde{\mathbf{n}} \in \partial \tilde{A}} \pi(\tilde{\mathbf{n}}) \tilde{q}(\tilde{\mathbf{n}}, A^c)}{\sum_{\tilde{\mathbf{n}} \in \partial \hat{A}} \pi(\tilde{\mathbf{n}}) \hat{q}(\tilde{\mathbf{n}}, A^c)} \geq \frac{\tilde{q}^{\min}}{\hat{q}^{\max}} \cdot \frac{\sum_{\tilde{\mathbf{n}} \in \partial \tilde{A}} \pi(\tilde{\mathbf{n}})}{\sum_{\tilde{\mathbf{n}} \in \partial \hat{A}} \pi(\tilde{\mathbf{n}})}.$$

Therefore, in order to show (10) it is enough to check that for any  $A \subset \tilde{\mathbb{E}}$ , such that  $\pi(A) > 0$

$$\infty > \frac{\sum_{\tilde{\mathbf{n}} \in \partial \hat{A}} \pi(\tilde{\mathbf{n}})}{\sum_{\tilde{\mathbf{n}} \in \partial \tilde{A}} \pi(\tilde{\mathbf{n}})} > 0. \quad (11)$$

Let us examine the difference between  $\pi(\tilde{\mathbf{n}})$  and  $\pi(\tilde{\mathbf{n}}')$ , where  $\tilde{\mathbf{n}}'$  and  $\tilde{\mathbf{n}}$  are different on each coordinate by not more than 1, and both  $\tilde{\mathbf{n}}$  and  $\tilde{\mathbf{n}}'$  have the same set of broken nodes  $I$ .

Recall from (5) that for  $\tilde{\mathbf{n}} = (D, n_1, \dots, n_m) \in \mathcal{P}(M) \times \mathbb{Z}_+^m$  we have:

$$\pi(\tilde{\mathbf{n}}) = \pi(D, n_1, \dots, n_m) = \frac{1}{C} \frac{\psi(D)}{\phi(D)} \prod_{i=1}^m \pi_i(n_i), \quad \text{where } \pi_i(n_i) := \frac{1}{C_i} \frac{\lambda_i^{n_i}}{\prod_{y=1}^{n_i} \mu_i(y)}.$$

For  $n_i \geq 1$ ,

$$\pi_i(n_i + 1) = \frac{1}{C_i} \frac{\lambda_i^{n_i+1}}{\prod_{y=1}^{n_i+1} \mu_i(y)} = \pi_i(n_i) \frac{\lambda_i}{\mu_i(n_i + 1)}$$

and

$$\pi_i(n_i - 1) = \frac{1}{C_i} \frac{\lambda_i^{n_i-1}}{\prod_{y=1}^{n_i-1} \mu_i(y)} = \pi_i(n_i) \frac{\lambda_i^{-1}}{\frac{1}{\mu_i(n_i)}},$$

thus we have bounds using  $\underline{\mu}_j := \inf_n \mu_j(n) > 0$ , and  $\bar{\mu}_j := \sup_n \mu_j(n) < \infty$ , of the form

$$\begin{aligned} \left(\frac{\lambda_i}{\underline{\mu}_i}\right) \pi_i(n_i) &\leq \pi_i(n_i + 1) \leq \pi_i(n_i) \left(\frac{\lambda_i}{\underline{\mu}_i}\right), \\ \left(\frac{\lambda_i}{\underline{\mu}_i}\right)^{-1} \pi_i(n_i) &\leq \pi_i(n_i - 1) \leq \pi_i(n_i) \left(\frac{\lambda_i}{\underline{\mu}_i}\right)^{-1}. \end{aligned}$$

For

$$d_i = \max \left( \left(\frac{\lambda_i}{\bar{\mu}_i}\right), \left(\frac{\lambda_i}{\bar{\mu}_i}\right)^{-1}, \left(\frac{\lambda_i}{\underline{\mu}_i}\right), \left(\frac{\lambda_i}{\underline{\mu}_i}\right)^{-1} \right) \quad \text{and} \quad d = \prod_{i=1}^m d_i, \quad (12)$$

we have

$$\left(\frac{1}{d_i}\right) \pi_i(n_i) \leq \pi_i(n_i \pm 1) \leq d_i \pi_i(n_i).$$

Thus, if we take  $\tilde{\mathbf{n}}$  and  $\tilde{\mathbf{n}}'$  which may differ by at most  $\pm 1$  at each coordinate, we obtain

$$\frac{1}{d} \pi(\tilde{\mathbf{n}}') \leq \pi(\tilde{\mathbf{n}}) \leq d \pi(\tilde{\mathbf{n}}'). \quad (13)$$

It is possible that there are states which are in  $\partial \tilde{A}$  and not in  $\partial \hat{A}$ , and also such that they are in  $\partial \hat{A}$  but not in  $\partial \tilde{A}$ . For  $\mathbf{z} \in \partial \tilde{A} \setminus \partial \hat{A}$ , there exists some  $\mathbf{y}_{\mathbf{z}} \in A^c$  such that process with kernel  $\hat{q}$  cannot move from  $\mathbf{z}$  to  $\mathbf{y}_{\mathbf{z}}$  in one step, but process with kernel  $\tilde{q}$  can. State  $\mathbf{y}_{\mathbf{z}}$  must be of the form  $\mathbf{y}_{\mathbf{z}} = T_{ij}\mathbf{z}$ , and it cannot be of the form  $\mathbf{y}_{\mathbf{z}} = T^I\mathbf{z}$  or  $\mathbf{y}_{\mathbf{z}} = T_H\mathbf{z}$ , since the change of the

set of broken nodes in the state  $\mathbf{z}$  is always possible for both processes. Moreover, it is neither of the form  $\mathbf{y}_{\mathbf{z}} = T_{i0}\mathbf{z}$  nor of the form  $\mathbf{y}_{\mathbf{z}} = T_{0j}\mathbf{z}$ , because both processes can move in these directions in one step. There exists however a two step path  $T_{i0}, T_{0j}$  such that  $\mathbf{z}' = T_{i0}\mathbf{z} \in A$  and  $\mathbf{y}_{\mathbf{z}} = T_{0j}\mathbf{z}' \in A^c$ , and then either  $\mathbf{z}' \in \partial\hat{A} \setminus \partial\tilde{A}$  or  $\mathbf{z}' \in \partial\hat{A} \cap \partial\tilde{A}$ . Because each  $\mathbf{z} \in \partial\tilde{A} \setminus \partial\hat{A}$  must be a *corner point*, it is possible to select different points  $\mathbf{z}' \in (\partial\hat{A} \setminus \partial\tilde{A}) \cup (\partial\hat{A} \cap \partial\tilde{A})$  for different points  $\mathbf{z}$ .

Thus using (13) we can give a rough bound

$$\begin{aligned}
\frac{\sum_{\tilde{\mathbf{n}} \in \partial\hat{A}} \pi(\tilde{\mathbf{n}})}{\sum_{\tilde{\mathbf{n}} \in \partial\tilde{A}} \pi(\tilde{\mathbf{n}})} &= \frac{\sum_{\tilde{\mathbf{n}} \in \partial\hat{A} \cap \partial\tilde{A}} \pi(\tilde{\mathbf{n}}) + \sum_{\tilde{\mathbf{n}} \in \partial\hat{A} \setminus \partial\tilde{A}} \pi(\tilde{\mathbf{n}})}{\sum_{\tilde{\mathbf{n}} \in \partial\hat{A} \cap \partial\tilde{A}} \pi(\tilde{\mathbf{n}}) + \sum_{\tilde{\mathbf{n}} \in \partial\tilde{A} \setminus \partial\hat{A}} \pi(\tilde{\mathbf{n}})} \\
&\geq \frac{\sum_{\tilde{\mathbf{n}} \in \partial\hat{A} \cap \partial\tilde{A}} \pi(\tilde{\mathbf{n}}) + \sum_{\tilde{\mathbf{n}} \in \partial\hat{A} \setminus \partial\tilde{A}} \pi(\tilde{\mathbf{n}})}{\sum_{\tilde{\mathbf{n}} \in \partial\hat{A} \cap \partial\tilde{A}} \pi(\tilde{\mathbf{n}}) + d \left( \sum_{\tilde{\mathbf{n}} \in \partial\hat{A} \cap \partial\tilde{A}} \pi(\tilde{\mathbf{n}}) + \sum_{\tilde{\mathbf{n}} \in \partial\hat{A} \setminus \partial\tilde{A}} \pi(\tilde{\mathbf{n}}) \right)} \\
&\geq \frac{\sum_{\tilde{\mathbf{n}} \in \partial\hat{A} \cap \partial\tilde{A}} \pi(\tilde{\mathbf{n}}) + \sum_{\tilde{\mathbf{n}} \in \partial\hat{A} \setminus \partial\tilde{A}} \pi(\tilde{\mathbf{n}})}{(d+1) \left( \sum_{\tilde{\mathbf{n}} \in \partial\hat{A} \cap \partial\tilde{A}} \pi(\tilde{\mathbf{n}}) + \sum_{\tilde{\mathbf{n}} \in \partial\hat{A} \setminus \partial\tilde{A}} \pi(\tilde{\mathbf{n}}) \right)} = \frac{1}{d+1}.
\end{aligned}$$

Fix  $\mathbf{x} \in \partial\hat{A} \setminus \partial\tilde{A}$ . Then there exists some  $\mathbf{y}_{\mathbf{x}} \in A^c$  such that original process with the kernel  $\tilde{q}$  cannot move there in one step, but the process with the kernel  $\hat{q}$  can. State  $\mathbf{y}_{\mathbf{x}}$  must be of the form  $\mathbf{y}_{\mathbf{x}} = T_{0i_0}\mathbf{x}$  or  $\mathbf{y}_{\mathbf{x}} = T_{j_00}\mathbf{x}$ , it cannot be of the form  $\mathbf{y}_{\mathbf{x}} = T^I\mathbf{y}$  or  $\mathbf{y}_{\mathbf{x}} = T_H\mathbf{y}$ , as previously. Because for a selected location in the state space, for the network process, at least one transition of the form  $T_{0k}$  must be possible, and at least one transition of the form  $T_{l0}$  must be possible (for some  $k, l$ ), and for the coordinates (nodes) where a transition from outside is not possible, there must be a migration transition into it ( $T_{s,i_0}$ , say), there exists path of length at most  $m$  such that process with kernel  $\tilde{q}$  gets along this path to  $\mathbf{y}_{\mathbf{x}}$ . The path consists only of transitions of the form  $T_{0i}, T_{j0}, T_{ij}$ , and at each intermediate step the coordinatewise distance to  $\mathbf{y}_{\mathbf{x}}$  is not larger than 1 on each coordinate. Further, there exists a state  $\mathbf{x}'$  on this path such that  $\mathbf{x}' \in \partial\hat{A} \cap \partial\tilde{A}$ , namely at the location where the transition  $T_{0k}$  is possible. Moreover the function  $\mathbf{x} \rightarrow \mathbf{x}'$  can be defined as 1-1. Thus, using (13), we again give a rough bound

$$\frac{\sum_{\tilde{\mathbf{n}} \in \partial\hat{A}} \pi(\tilde{\mathbf{n}})}{\sum_{\tilde{\mathbf{n}} \in \partial\tilde{A}} \pi(\tilde{\mathbf{n}})} = \frac{\sum_{\tilde{\mathbf{n}} \in \partial\hat{A} \cap \partial\tilde{A}} \pi(\tilde{\mathbf{n}}) + \sum_{\tilde{\mathbf{n}} \in \partial\hat{A} \setminus \partial\tilde{A}} \pi(\tilde{\mathbf{n}})}{\sum_{\tilde{\mathbf{n}} \in \partial\hat{A} \cap \partial\tilde{A}} \pi(\tilde{\mathbf{n}}) + \sum_{\tilde{\mathbf{n}} \in \partial\tilde{A} \setminus \partial\hat{A}} \pi(\tilde{\mathbf{n}})}$$

$$\begin{aligned}
& \leq \frac{\sum_{\tilde{\mathbf{n}} \in \partial \hat{A} \cap \partial \tilde{A}} \pi(\tilde{\mathbf{n}}) + d \left( \sum_{\tilde{\mathbf{n}} \in \partial \hat{A} \cap \partial \tilde{A}} \pi(\tilde{\mathbf{n}}) + \sum_{x \in \partial \tilde{A} \setminus \partial \hat{A}} \pi(\tilde{\mathbf{n}}) \right)}{\sum_{\tilde{\mathbf{n}} \in \partial \hat{A} \cap \partial \tilde{A}} \pi(\tilde{\mathbf{n}}) + \sum_{\tilde{\mathbf{n}} \in \partial \tilde{A} \setminus \partial \hat{A}} \pi(\tilde{\mathbf{n}})} \\
& \leq \frac{(1+d) \left( \sum_{\tilde{\mathbf{n}} \in \partial \hat{A} \cap \partial \tilde{A}} \pi(\tilde{\mathbf{n}}) + \sum_{\tilde{\mathbf{n}} \in \partial \tilde{A} \setminus \partial \hat{A}} \pi(\tilde{\mathbf{n}}) \right)}{\sum_{\tilde{\mathbf{n}} \in \partial \hat{A} \cap \partial \tilde{A}} \pi(\tilde{\mathbf{n}}) + \sum_{\tilde{\mathbf{n}} \in \partial \tilde{A} \setminus \partial \hat{A}} \pi(\tilde{\mathbf{n}})} = 1 + d < \infty.
\end{aligned}$$

This finishes the proof of Lemma 3.1.

The statement of Theorem 3.2 is a direct consequence of the above lemma and Proposition 3.4.

### 3.3 Speed of convergence

Before formulating a result about speed of convergence for networks we recall from Mu-Fa Chen [5] some useful theorems. For a general picture showing ergodic results for Markov processes see this reference. For a Markov process governed by semigroup  $(P_t)$ , with generator  $\mathbf{Q}$ , the classical *variational formula* can be written as  $\text{Gap}(\mathbf{Q}) = \inf\{-\langle f, \mathbf{Q}f \rangle_\pi : \pi(f) = 0, \|f\|_2 = 1\}$ , where  $\langle f, g \rangle_\pi = \int f(x)g(x)\pi(dx)$ ,  $\pi$  is invariant for  $(P_t)$ , and  $\pi(f) = \int f(x)\pi(dx)$ ,  $\text{Var}_\pi(f) = \pi(f^2) - (\pi(f))^2$ . Let  $f \in L^2(\mathbb{E}, \pi)$ . Then  $C = \text{Gap}(\mathbf{Q})^{-1}$  is optimal in:

*Poincare inequality*:  $\text{Var}_\pi(f) \leq C \cdot \langle f, -\mathbf{Q}f \rangle_\pi$ .

It is known that the above inequality is related to the following convergence (see e.g. Mu-Fa Chen [5], Theorem 1.8.).

**Theorem 3.6.** *Poincare inequality holds if and only if Markov process converges according to  $L_2(\mathbb{E}, \pi)$ - exponential convergence:*

*for all  $f \in L^2(\mathbb{E}, \pi)$*

$$\|P_t f - \pi(f)\|_2^2 = \text{Var}(P_t f) \leq \text{Var}(f) \exp(-2\text{Gap}(\mathbf{Q})t), \quad t > 0,$$

Dual convergences for distributions one defines by (here the subscript *tv* denotes convergence in the total variation norm):

*ergodicity*: for each  $\mathbf{e} \in \mathbb{E}$

$$\|\delta_{\mathbf{e}} P_t - \pi\|_{tv} \rightarrow 0, \quad t \rightarrow \infty,$$

*exponential ergodicity*: for each  $\mathbf{e} \in \mathbb{E}$  there exists  $C(\mathbf{e}) > 0$  such that for some  $\alpha > 0$

$$\|\delta_{\mathbf{e}} P_t - \pi\|_{tv} \leq C(\mathbf{e}) \exp(-\alpha t), \quad t > 0.$$

The above statements are related in the following way (see e.g. Mu-Fa Chen [5], Theorem 1.9.)

**Theorem 3.7.** Suppose that  $\mathbb{E}$  is countable and  $P_t$  reversible. Then for all  $f \in L^2(\mathbb{E}, \pi)$

$$\|P_t f - \pi(f)\|_2^2 = \text{Var}(P_t f) \leq \text{Var}(f) \exp(-2\text{Gap}(\mathbf{Q})t), \quad t > 0,$$

if and only if for each  $\mathbf{e} \in \mathbb{E}$  there exists  $C(\mathbf{e}) > 0$  such that

$$\|\delta_{\mathbf{e}} P_t - \pi\|_{tv} \leq C(\mathbf{e}) \exp(-\text{Gap}(\mathbf{Q})t), \quad t > 0.$$

For non-reversible cases the exponent  $\alpha$  in the exponential ergodicity definition can be different from  $\text{Gap}(\mathbf{Q})$ .

As a consequence of Mu-Fa Chen's [5], Theorem 1.9. and Theorem 8.8 we obtain

**Proposition 3.8.** Let  $\tilde{\mathbf{X}}$  be unreliable Jackson network following the RS-RD-BLOCKING, with generator  $\tilde{\mathbf{Q}}$ , given by (4), and the corresponding transition semigroup  $(\tilde{P}_t)$ . Suppose the routing matrix  $R$  is reversible, and  $R^k > 0$  for some  $k \geq 1$ .

If  $\pi_i$  is strongly light-tailed, for each  $i = 1, \dots, m$ , then equivalently

(i) for all  $f \in L^2(\tilde{\mathbb{E}}, \pi)$

$$\|\tilde{P}_t f - \pi(f)\|_2 \leq e^{-\text{Gap}(\tilde{\mathbf{Q}})t} \|f - \pi(f)\|_2, \quad t > 0,$$

(ii) for each  $\mathbf{e} \in \tilde{\mathbb{E}}$  there exists  $C(\mathbf{e}) > 0$  such that

$$\|\delta_{\mathbf{e}} \tilde{P}_t - \pi\|_{tv} \leq C(\mathbf{e}) e^{-\text{Gap}(\tilde{\mathbf{Q}})t}, \quad t > 0.$$

**Remark.** For the classical Jackson networks, i.e. networks without breakdowns and repairs ( $\psi(\emptyset) = \phi(\emptyset) = 1, \psi(\bar{D}) = \phi(\bar{D}) = 0$  for  $\bar{D} \neq \emptyset$ ) the reversibility assumption on the routing matrix  $R$  can be relaxed in order to obtain the implication (i)  $\Rightarrow$  (ii).

## 4 Discussion of the assumptions

We formulated the results on the positivity of the spectral gap and on the convergence to stationarity in terms of the discrete hazard function of the stationary distribution. We assumed that the hazard function is separated from zero and called this property exponential light-tailness. For queueing networks it would be however more reasonable to formulate the assumptions in terms of the parameters of the network. Therefore we shall now introduce so called equilibrium rates for discrete distributions and use them to reformulate our assumptions and to connect them with the service rates in the system.

## 4.1 Discrete hazard rates and equilibrium rates

For a non-negative random variable  $X \in \mathbb{Z}_+$ , with probability function  $p(k) = P(X = k)$ , such that for any  $k \in \mathbb{Z}_+$ ,  $P(X = k) > 0$ , recall that one defines the *equilibrium rate* function by

$$e_p(k) = \begin{cases} \frac{p(k-1)}{p(k)} & \text{if } k \geq 1, \\ 0 & \text{if } k = 0, \end{cases}$$

and the *hazard function* by

$$h_p(k) = \frac{p(k)}{\bar{F}(k-1)}, \quad k \geq 0, \quad (14)$$

where  $\bar{F}(k) = P(X > k)$ . Distribution (or random variable) is *strongly light-tailed* if there exists  $\epsilon > 0$  such that  $\inf_k h_p(k) > \epsilon$ .

Since the equilibrium rate function  $(e_p(k), k \geq 1)$  uniquely determines the probability function  $(p(k), k \geq 1)$ , the hazard function  $(h_p(k), k \geq 0)$  uniquely determines the corresponding distribution. It is therefore possible to express light-tailness in terms of equilibrium rates. The following formulas connect both functions

$$e_p(k+1) = \frac{h_p(k)}{h_p(k+1)} \frac{1}{(1 - h_p(k))}, \quad k \geq 0 \quad (15)$$

and

$$h_p(k) = \frac{1}{1 + \sum_{j=k+1}^{\infty} \frac{1}{e_p(k+1) \cdots e_p(j)}}, \quad k \geq 0. \quad (16)$$

It is worth mentioning that each discrete distribution with the support  $\mathbb{Z}_+$  can appear as the stationary distribution for a birth and death process with constant birth rates. Moreover the existence of the spectral gap can be characterized directly using the birth and death rates.

**Lemma 4.1.** *Consider  $\{p(k)\}_{k \geq 0}$  an arbitrary probability function on  $\mathbb{Z}_+$ , such that  $p(k) > 0, k \geq 0$ , with the corresponding equilibrium rate  $e_p(k), k \geq 0$ . Then for each birth and death process  $\mathbf{X}$  with fixed  $\lambda(k) \equiv \lambda > 0, k \geq 0$ , and*

$$\mu(k) = \lambda \cdot e_p(k), \quad k \geq 0,$$

(i) *the stationary distribution of  $\mathbf{X}$  is equal to  $p(k), k \geq 0$ ,*

(ii) *the spectral gap for  $\mathbf{X}$  exists if and only if*

$$\inf_k \frac{1}{1 + \sum_{j=k+1}^{\infty} \frac{\lambda^{j-k}}{\mu(k+1) \cdots \mu(j)}} > 0$$

Proof. (i). For the stationary distribution  $\pi$  of the birth death process we have

$$\pi(i)/\pi(0) = \frac{\lambda \cdots \lambda}{\lambda \cdots \lambda \frac{p(0)}{p(1)} \frac{p(1)}{p(2)} \cdots \frac{p(i-1)}{p(i)}} = \frac{\lambda^i}{\lambda^i \frac{p(0)}{p(i)}} = p(i)/p(0), \quad i \geq 1.$$

Thus we have (i).

The conclusion in (ii) follows directly from (16), and Proposition 3.4.  $\square$

Neither  $h_p(k)$  nor  $e_p(k)$  have to be convergent as  $k \rightarrow \infty$ . However from (15), (16) we obtain a connection between these limits if they exist and are finite.

**Lemma 4.2.** *Consider  $\{p(k)\}_{k \geq 0}$  an arbitrary probability function on  $\mathbb{Z}_+$ , such that  $p(k) > 0, k \geq 0$ , with the corresponding equilibrium rate  $e_p(k), k \geq 0$ . Then*

*$h_p = \lim_{k \rightarrow \infty} h_p(k)$  exists and  $h_p \in (0, 1)$  if and only if  $e_p = \lim_{k \rightarrow \infty} e_p(k)$  exists and  $e_p \in (1, \infty)$ .*

*In this case*

$$h_p = 1 - \frac{1}{e_p}.$$

From the above lemma and in addition observing that non-decreasing  $\mu(k)$  as a function of  $k$  corresponds to the hazard function which is a non-decreasing function we have two situations when the spectral gap is positive for birth and death processes with constant birth rates.

**Corollary 4.3.** *Suppose for a birth and death process  $\mathbf{X}$  with fixed  $\lambda(k) \equiv \lambda > 0, k \geq 0$ , and  $\mu(k), k \geq 0$ , that  $\lim_{k \rightarrow \infty} \mu(k)/\lambda$  exists and is strictly greater than 1. Then the stationary distribution  $\pi$  for  $\mathbf{X}$  is strongly light-tailed and the spectral gap for  $\mathbf{X}$  exists.*

*Moreover, if  $\mu(k)$  is non-decreasing as a function of  $k$  then*

$$\text{Gap}(\mathbf{Q}) \geq \frac{\lambda \epsilon^2}{2(1 - \epsilon)(2 - \epsilon)},$$

*for  $\epsilon = \pi(0) = (1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{\prod_{y=1}^n \mu(y)})^{-1} > 0$ .*

**Corollary 4.4.** *Let  $\tilde{\mathbf{X}}$  be unreliable Jackson network process following the RS-RD-BLOCKING, with the infinitesimal generator  $\tilde{\mathbf{Q}}$ . Suppose that  $\tilde{\mathbf{Q}}$  is bounded and the minimal service intensity  $\underline{\mu} > 0$ . If the routing matrix  $R$  is reversible, and regular then  $\text{Gap}(\tilde{\mathbf{Q}}) > 0$  if and only if for each  $i = 1, \dots, m$ ,*

$$\inf_k \frac{1}{1 + \sum_{j=k+1}^{\infty} \frac{\lambda_i^{j-k}}{\mu_i(k+1) \dots \mu_i(j)}} > 0.$$

*In particular, if for each  $i = 1, \dots, m$ ,  $\lim_{k \rightarrow \infty} \mu_i(k)/\lambda_i$  exists and is strictly greater than 1, then  $\text{Gap}(\tilde{\mathbf{Q}}) > 0$ .*

We shall now indicate relations between discrete hazard functions and usual (continuous) hazard functions, and show that strong light-tailness implies the usual light-tailness. The *total hazard function*  $H_p$  is defined for all  $x \geq 0$  by

$$H_p(x) = -\log \bar{F}(x).$$

Note that for natural arguments  $k \geq 0$

$$H_p(k) = -\log \prod_{j=0}^k (1 - h_p(j)), \quad (17)$$

and for arbitrary  $x \geq 0$

$$H_p(x) = -\log \prod_{j=0}^{\lfloor x \rfloor} (1 - h_p(j)) = \sum_{j=0}^{\lfloor x \rfloor} \log \left( \frac{1}{1 - h_p(j)} \right), \quad (18)$$

where  $\lfloor x \rfloor$  denotes the integer part of  $x$ .

Note that each sequence  $h(i) \in (0, 1)$ ,  $i = 0, 1, \dots$  defines by the above formula a discrete distribution function  $F$  with the support  $\{0, 1, 2, \dots\}$ . An arbitrary distribution  $F$  with its support contained in  $[0, \infty)$  is light-tailed if  $\int_0^\infty e^{sx} dF(x) < \infty$ . It is known (see e.g. Rolski *et al.* [21], Th. 2.3.1) that

$$\liminf_{x \rightarrow \infty} -\frac{1}{x} \log(\bar{F}(x)) > 0$$

implies that  $F$  is light-tailed.

**Lemma 4.5.** *Consider a random variable  $X \in \mathbb{Z}_+$ , with probability function  $p(k) = P(X = k)$ , such that for any  $k \in \mathbb{Z}_+$ ,  $P(X = k) > 0$ , and  $p$  is strongly light-tailed. Then it is light-tailed in the usual sense.*

Proof. Note that

$$\frac{H_p(x)}{x} \geq \frac{H_p(\lfloor x \rfloor)}{\lfloor x \rfloor + 1},$$

for all  $x \geq 0$ , therefore

$$\inf_n \frac{H_p(n)}{n+1} > 0 \Rightarrow \liminf_{x \rightarrow \infty} \frac{H_p(x)}{x} > 0. \quad (19)$$

From the exponential light-tailness we have for all  $j$ ,  $\log(\frac{1}{1-h_p(j)}) > \log(\frac{1}{1-\epsilon})$ , and hence from (18)

$$\frac{H_p(n)}{n+1} > \log \left( \frac{1}{1-\epsilon} \right) > 0,$$

which from (19) implies that  $F_p$  is light-tailed. □

In order to see that exponential light-tailness is strictly a stronger notion than light-tailness for discrete distributions we give a simple example. This example shows at the same time that there exists a birth and death process having its rate of convergence to stationarity not exponentially fast, but having its stationary distribution light-tailed.

**Example 4.6.** Let us take as  $p$  the distribution which corresponds to the hazard function  $h_p$  given by  $h_p(1) = 1/2$ ,

$$h_p(k) = \begin{cases} 1/k & \text{if } k = 2n+1, \quad n \geq 1, \\ 1/2 & \text{if } k = 2n, \quad n \geq 0. \end{cases}$$

This distribution is not strongly light-tailed since  $\inf_k h_p(k) = 0$ . However, for each natural  $n$ ,  $\lim_{n \rightarrow \infty} \frac{H_p(2n+1)}{2n+2} = \lim_{n \rightarrow \infty} \frac{H_p(2n)}{2n+1} = \log(2)/2 > 0$ , and from (19) we obtain that  $p$  is light-tailed. From Lemma 4.2 we can define the corresponding birth and death process with its stationary distribution equal to  $p$ , and having constant birth rates. The rate of convergence to stationarity for this process is not exponentially fast, however its stationary distribution is light-tailed.

## 4.2 Bounds on the spectral gap

From the proof of Lemma 3.1 we can get bounds on  $Gap(\tilde{\mathbf{Q}})$ . We have  $\kappa_1 \geq v_1 \kappa_2$ , where  $\kappa_1$  and  $\kappa_2$  are the Cheeger's constants of  $\tilde{\mathbf{Q}}$  and  $\hat{\mathbf{Q}}$ , respectively. Using rough estimates from the proof of Lemma 3.1, we have

$$v_1 := \frac{\tilde{q}^{min}}{\hat{q}^{max}} \frac{1}{d+1}.$$

From Theorems 2.1 and 2.3 of Lawler and Sokal [18] and from Theorem 3.5 we have

$$Gap(\tilde{\mathbf{Q}}) \geq \frac{v_1^2}{8|\tilde{\mathbf{Q}}|} \left( \min_i Gap(\hat{\mathbf{Q}}_i) \right)^2. \quad (20)$$

By Proposition 3.4

$$\min_i Gap(\hat{\mathbf{Q}}) \geq \min_i \frac{\lambda_i \epsilon_i^2}{2(1 - \epsilon_i)(2 - \epsilon_i)}, \quad (21)$$

where  $\epsilon_i$  is a lower bound on the hazard function  $h_{\pi_i}$ .

For a Jackson network with constant service rates  $\mu_i(n) = \mu_i$  we have constant hazard function  $h_{\pi_i}(k) = 1 - \frac{\lambda_i}{\mu_i}$ , and

$$Gap(\tilde{\mathbf{Q}}) \geq \frac{v_1^2}{32|\tilde{\mathbf{Q}}|} \left( \min_i \frac{\mu_i \left(1 - \frac{\lambda_i}{\mu_i}\right)^2}{\left(1 + \frac{\lambda_i}{\mu_i}\right)} \right)^2.$$

Calculating the bound using this method is quite straightforward, provided the solution of the traffic equation is given. In Iscoe and McDonald [15], their method requires some computations for two intermediate birth and death processes. For large number of servers, it is far from being straightforward.

**Example 4.7.** We will give a bound for the spectral gap of the standard two-node Jackson network with parameters:

$$\lambda = 12, \quad \mu_1(n) = 20 - 2^{-n}, \mu_2(n) = 16 - 4^{-n}, \quad R = \begin{bmatrix} 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \\ 3/4 & 1/4 & 0 \end{bmatrix}$$

Solving traffic equation yields  $\lambda_1 = 48/7, \lambda_2 = 80/7$ . One can easily check, that system is stable. We have  $\underline{\mu}_1 = 19, \bar{\mu}_1 = 20, \underline{\mu}_2 = 15, \bar{\mu}_2 = 16$ , and  $\tilde{q}^{min} = 4, \hat{q}^{max} = 20$ . Further,  $d = d_1 d_2 = 49/12$ .

Using  $|\tilde{\mathbf{Q}}| = \lambda + \bar{\mu}_1 + \bar{\mu}_2 = 48$  and  $\nu_1 = \frac{4}{20}(1/(1 + 49/12)) = 60/61$  we have from (20)

$$Gap(\tilde{\mathbf{Q}}) \geq \frac{75}{29768} \left( \min_i Gap(\hat{\mathbf{Q}}_i) \right)^2.$$

Since  $\mu_1(n)$  and  $\mu_2(n)$  are non-decreasing, we can have bounds on  $Gap(\hat{\mathbf{Q}}_i), i = 1, 2$  from Corollary 4.3. Calculating  $\epsilon_i = (1 + \sum_{n=1}^{\infty} \frac{\lambda_i^n}{\prod_{y=1}^n \mu_i(y)})^{-1}$  for  $i = 1, 2$  we obtain  $\epsilon_1 = 0.6502, \epsilon_2 = 0.2818$ . This provides bounds:  $Gap(\hat{\mathbf{Q}}_1) \geq 4.7220$  and  $Gap(\hat{\mathbf{Q}}_2) \geq 0.7839$ .

Finally

$$Gap(\tilde{\mathbf{Q}}) \geq \frac{75}{29768} (0.7830)^2 = 0.001544667932.$$

# References

- [1] ADAN, I., FOLEY, R. D. AND McDONALD, D. R. (2009). Exact asymptotic for the stationary distribution of a Markov chain: a production model. *Queueing Systems* **62**, 311–344.
- [2] BACCELLI, F. AND FOSS, F. (1994). Ergodicity of Jackson-type queueing networks. *Queueing Systems* **17** 5–72.
- [3] BACCELLI, F., FOSS, F. AND MAIRESSE, J. (1996). Stationary ergodic Jackson networks: results and counter-examples. *Stochastic Networks, Theory and Applications*, Royal Statistical Society Lecture Notes 4, Oxford Science Publications.
- [4] CHANG, C.-S., THOMAS, J. A. AND KIANG, S.-H. (1994). On the stability of open networks: a unified approach by stochastic dominance. *Queueing Systems* **15**, 239–260.
- [5] CHEN, M.-F. (2005). *Eigenvalues, Inequalities, and Ergodic Theory*. Springer.
- [6] COOLEN-SCHRIJNER, P. AND VAN DOORN, E. A. (2002). The deviation matrix of a continuous-time Markov chain. *Probability in the Engineering and Informational Sciences* **16**(3), 351–366.
- [7] DADUNA, H. AND SZEKLI, R. Impact of routing on correlation strength in stationary queueing network processes. (2008). *J. Appl. Prob.* **45**, 846–878.
- [8] DIEKER, A. B. AND WARREN, J. (2009). Series Jackson Networks and non-crossing probabilities. [arXiv:0808.0513v3 \[math.PR\]](#).
- [9] DUPUIS, P. AND WANG, H. (2009). Importance sampling for Jackson networks. *Queueing Systems* **62**, 113–157.
- [10] FAYOLLE, G., MALYSHEV, V. A., MENSHIKOV, M. V. AND SIDORENKO, A. F. (1993). Lyapunov functions for Jackson networks. *Math. Operat. Res.* **18**(4), 916–927.
- [11] FOLEY, R. D. AND McDONALD, D. R. (2005). Large deviations of a modified Jackson network: stability and rough asymptotic. *Annals of Applied Probability*. **15**, 519–541.
- [12] GAMARNIK, D. AND MEYN, S. (2006). On Exponential Ergodicity of Multiclass Queueing Networks. [arXiv:math/0612544v1 \[math.PR\]](#).
- [13] HORDIJK, A. AND SPIEKSMAN, F. (1992). On ergodicity and recurrence properties of markov chain with an application to an open Jackson network. *Adv. Appl. Prob.* **24**, 343–376.
- [14] IGNATIOUK-ROBERT, I. (2000). Large Deviations of Jackson networks. *Ann. Prob.* **10**(3): 962–1001.
- [15] ISCOE, I. AND McDONALD, D. (1994). Asymptotics of exit times for markov jump processes ii: applications to Jackson networks. *Ann. Prob.* **22**, 2168–2182.
- [16] KOOLE, G. M. AND SPIEKSMAN, F. M. (2001). On deviation matrices for birth-death processes. *Probability in the Engineering and Informational Sciences* **15**(2), 239–258.
- [17] KROESE, D. P., SCHEINHARDT, W. R. W. AND TAYLOR, P. G. (2004). Spectral properties of the tandem Jackson network, seen as a quasi-birth-and-death process. *Ann. Prob.* **14**(4), 2057–2089.
- [18] LAWLER, G. F. AND SOKAL, A. D. (1988). Bounds on the  $L^2$  Spectrum for Markov Chains and Markov Processes: A Generalization of Cheeger’s Inequality. *Trans. Amer. Math. Soc.* **309**(2), 557–580.
- [19] LIGGETT, T. H. (1989). Exponential  $l_2$  convergence of attractive reversible nearest particle systems. *Ann. Prob.* **17**, 403–432.

- [20] LOREK, P. (2010). The exact asymptotic for the stationary distribution of some unreliable systems. Preprint, Wroclaw University (submitted) <http://www.mathstat.uottawa.ca/~plorek/>.
- [21] ROLSKI, T., SCHMIDLI, H., SCHMIDT, V. AND TEUGELS, J. (1999). *Stochastic Processes for Insurance and Finance*. John Wiley & Sons.
- [22] SAUER, C. (2006). *Stochastic Product From Networks with Unreliable Nodes: Analysis of Performance and Availability*. PhD thesis, Hamburg University.
- [23] SAUER, C. AND DADUNA, H. (2003). Availability formulas and performance measures for separable degradedable networks. *Economic Quality Control* **18**(2), 165–194.
- [24] E. VAN DOORN. (2002). Representations for the rate of convergence of birth-death processes. *Theory Probab. Math. Statist.* **65**, 37–43.